28/06/16

More about isogenies & cocycles

<u>Goal:</u> Describe ker and coker of

$$\mathcal{P}_b \xrightarrow{②} \mathcal{P}_b^1 \underset{j^1}{\xrightarrow{①}} \mathrm{Bun}_W^T(\tilde{X}_b) \xrightarrow{③} \mathrm{Prym}_\Lambda(\tilde{X}_b) \ !$$

$\left(\text{over } b \in B^\circ(k) = (B \setminus \Delta)(k) \right)$

Here

$$\begin{tikzcd}
& \tilde{X} \arrow[dd] \arrow[drr] & & \\
\tilde{X}_b \arrow[ur] \arrow[rrr] \arrow[dd] & & & \mathfrak{t}\otimes\mathcal{L} \arrow[dd, "\pi"] \\
& X \times B \arrow[drr] & & \\
X \arrow[ur, "b"] \arrow[rrr, "b"] & & & c_{\mathcal{L}} := \mathfrak{t}\otimes\mathcal{L}/W
\end{tikzcd}$$

"cameral cover"

1. Tautological equivariant structures on $\tilde{X}^\alpha$

Similar to the case $G = Gl_n$ from last time, we have:

<u>Prop. 1</u> There is a natural exact sequence

$$0 \longrightarrow \mathcal{P}^1 \xrightarrow{j^1} \mathrm{Bun}_T^W(\tilde{X}) \xrightarrow{+} \Big(\prod_{\alpha\in\Phi} \pi_*(T_\alpha, \tilde{X}^\alpha) \Big)^W$$

where $T_\alpha := T/(s_\alpha - 1)$ and $\tilde{X}^\alpha := \mathrm{Fix}(s_\alpha)$.

Proof.

Step 0: Def^n of r

For $(E, \gamma) \in \mathcal{B}un_T^W(\tilde{X})(S)$, $\alpha \in \Phi$,

have

$$s_\alpha(E) \overset{T}{\times} T_\alpha \Big|_{\tilde{X}^\alpha} \underset{\gamma_{s_\alpha}}{\overset{\tau}{\rightrightarrows}} E \overset{T}{\times} T_\alpha \Big|_{\tilde{X}^\alpha}$$

where τ is the tautological iso.

Put $$r(E,\gamma) := (\underbrace{\gamma_{s_\alpha} \circ \tau^{-1}}_{\in T_\alpha|_{\tilde{X}^\alpha_S}})_{\alpha \in \Phi}$$

Check: $r \circ j^1 = 0$, (easy)

using $j^1(\mathcal{M}) := \pi^*(\mathcal{M}) \overset{\pi^*(j^1)}{\times} T$ with canonical equivar. structure

Step 1: Suffices to show the following

Claim: (*) Any $(E, \gamma) \in \ker(r)(S)$ is étale-locally over X_S isomorphic to the trivial equivariant torsor $(T_{\tilde{X}_S}, \gamma_{can})$.

Indeed:

$(*) \Longrightarrow \quad \mathcal{M} := (\pi_* E)^W$ locally isomorphic to $\pi_*(T_{\tilde{X}_S})^W = \mathcal{J}^1_{\mathcal{X}}$

& any two such iso's induced by $(*)$ differ only by a section of $\mathcal{J}^1_{\mathcal{X}}$

$\Longrightarrow \quad \mathcal{M} \in \mathcal{P}^1(S)$ is a $\mathcal{J}^1_{\mathcal{X}}$-torsor on X_S

Local triviality easily implies $(E, \gamma) \simeq j^1(\mathcal{M})$

and so $\mathcal{P}^1 \simeq \ker(+)$.

<u>Step 2</u>: Reduce claim $(*)$ to totally ramified case:

$$\begin{tikzcd}
\tilde{x}_0 \in \tilde{X}_S \arrow[r, "\tilde{\iota}"] \arrow[d, "\pi"] & \mathfrak{t} \otimes \mathcal{L} \arrow[d] \\
x_0 \in X_S \arrow[r, "\iota"] & \mathfrak{t} \otimes \mathcal{L}/W
\end{tikzcd}$$

x_0 closed pt.

Locally on X_S, wlog $\mathcal{L} \simeq \mathcal{O}$ trivial.

Fix $\tilde{x}_0 \longmapsto x_0$

& $\Phi_0 \subseteq \Phi$

s.th. $\tilde{\iota}(\tilde{x}_0) \in \ker(d\alpha)$ $\forall \alpha \in \Phi_0$

but $\notin \ker(d\beta)$ for $\beta \in \Phi \setminus \Phi_0$.

Put $W_0 := \langle s_\alpha \mid \alpha \in \Phi_0 \rangle \subset W$,

$$\mathfrak{t}_0 := \mathfrak{t} \setminus \bigcup_{\beta \in \Phi \setminus \Phi_0} \ker(d\beta),$$

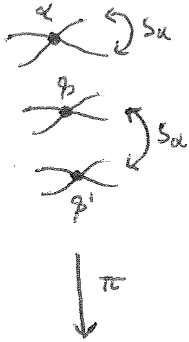
$$\tilde{X}_{s,0} := \tilde{X}_s \times_{\mathfrak{t}} \mathfrak{t}_0 \quad \text{and} \quad X_{s,0} := \tilde{X}_{s,0}/W_0.$$

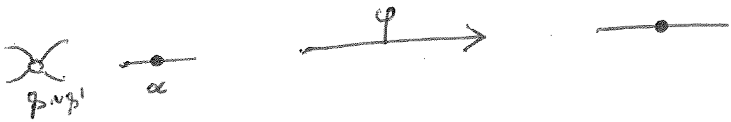
$$\left(= \tilde{X}_s \setminus \bigcup_{\beta \notin \Phi_0} \tilde{X}_s^\beta \right)$$

$\Rightarrow$ Get Cartesian diagram

$$\begin{tikzcd} W \overset{W_0}{\times} \tilde{X}_{s,0} \arrow[r, "(w,\tilde{x}) \mapsto w \cdot \tilde{x}"] \arrow[d, "\pi_0"] \arrow[dr, phantom, "\lrcorner"] & \tilde{X}_s \arrow[d, "\pi"] \\ X_{s,0} \arrow[r, "\varphi"'] & X_s \end{tikzcd}$$

$(w, \tilde{x}) \mapsto (\tilde{x} \bmod W_0)$





where

- φ is étale at $\pi_0(\tilde{x}_0) =: \tilde{y}_0$
- $\tilde{X}_{s,0} \longrightarrow X_{s,0}$ is totally ramified over $\tilde{y}_0$.

Descent: $\quad \mathrm{Bun}_T^W(\tilde{X}_s)^{\tau=0} \simeq \mathrm{Bun}_T^{W_0}(\tilde{X}_{s,0})^{\tau=0}$

$\Longrightarrow$ can replace $\tilde{X}_s \to X_s$ by $\tilde{X}_{s,0} \to X_{s,0}$.

<u>Step 3</u>: Proof of $(*)$ in the totally ramified case:

$$\pi^{-1}(x_0) = \{\tilde{x}_0\}$$

Localize:

$$\begin{array}{ccc} \operatorname{Spec} \tilde{\mathcal{O}} & \longrightarrow & \tilde{X}_S \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathcal{O} & \longrightarrow & X_S \end{array} \qquad \begin{array}{l} \tilde{\mathcal{O}} := \mathcal{O}_{\tilde{X}_S, \tilde{x}_0} \\ \\ \mathcal{O} := \mathcal{O}_{X_S, x_0} \end{array}$$

Pick trivialization of E near $\tilde{x}_0$,
then equivariant structure is given by

$$[\gamma] \in H^1(W, T(\tilde{\mathcal{O}})).$$

Define $\mathcal{U}$ by $\quad 1 \to \mathcal{U} \to T(\tilde{\mathcal{O}}) \xrightarrow{\ ev\ } T(k) \to 1,$

then

$$\underbrace{H^1(W,\mathcal{U})} \longrightarrow H^1(W, T(\tilde{\mathcal{O}})) \longrightarrow H^1(W, T(k))$$

$$[\gamma] \in H^1(W, T(\tilde{\mathcal{O}})) \longmapsto 0 \in H^1(W, T(k))$$

since $(E,\gamma) \in \ker(*)$

- $H^1(W,\mathcal{U})$ annihilated by $|W|$
- but $\mathcal{U} \xrightarrow{|W|} \mathcal{U}$ is an iso since $\mathcal{U}$ has filtration w/ all subquotients $\mathbb{F}_p$-vector spaces & $p \nmid |W|$
- hence $H^1(W,\mathcal{U}) = 0$

$$\Longrightarrow [\gamma] = 0$$

$\square$

2. Coroot conditions on $\tilde{X}^\alpha$

Recall: roots $\alpha \in \Phi \subset \mathrm{Hom}(T, \mathbb{G}_m)$

$\updownarrow$

coroots $\check{\alpha} \in \check{\Phi} \subset \mathrm{Hom}(\mathbb{G}_m, T)$

sth $\alpha \circ \check{\alpha} = [2] : \mathbb{G}_m \to \mathbb{G}_m$.

Def. $\mu_{\check{\alpha}} := \ker(\check{\alpha}) \subseteq \{\pm 1\}$.

Remark 2. If $G^{der} = [G, G]$ is simply connected or if the Dynkin diagram of G has no factor B_n with $n \geq 1$, then $\mu_{\check{\alpha}} = \mathbb{1}$ $\forall \alpha$.

Proof. In these cases, $\exists\, p \in \mathrm{Hom}(T, \mathbb{G}_m)$ sth $p \circ \check{\alpha} = id$. $\square$

Prop. 3 There is a natural exact sequence

$$0 \longrightarrow \Big(\prod_{\alpha \in \Phi} \pi_*(\mu_{\check{\alpha}}, \tilde{X}^\alpha)\Big)^W \longrightarrow \mathcal{P} \longrightarrow \mathcal{P}^1 \longrightarrow 0$$

Proof. Use $0 \to \mathcal{J}_L \to \mathcal{J}_L^1 \to \Big(\prod_{\alpha \in \Phi} \pi_*(\mu_{\check{\alpha}}, \tilde{X}^\alpha)\Big)^W \to 0$

& formalism of Picard stacks, or explicit description below. $\square$

Note: For $(E,\gamma) \in \mathcal{B}un_T^W(\tilde{X})(S)$,

$\exists$ natural iso φ_α:

$$\begin{tikzcd}
(E \overset{T,\check{\alpha}\circ\alpha}{\times} T)|_{\tilde{X}^\alpha} \arrow[rr, "\sim", "\varphi_\alpha"'] \arrow[d, "\boxed{\check{\alpha}\circ\alpha = 1 - s_\alpha \text{ on } T}", "\wr"'] & & T_{\tilde{X}^\alpha} \\
(E \otimes s_\alpha(E)^{-1})|_{\tilde{X}^\alpha} \arrow[rr, "\sim", "\gamma_\alpha"'] & & (E \otimes E^{-1})|_{\tilde{X}^\alpha} \arrow[u, "\wr"']
\end{tikzcd}$$

Here $(E,\gamma) \in \ker(*)$ iff these φ_α come from trivializations

$$c_\alpha : (E \overset{T,\alpha}{\times} \mathbb{G}_m)|_{\tilde{X}^\alpha} \xrightarrow{\sim} \mathbb{G}_{m,\tilde{X}^\alpha}$$

compatible with W-equivariant structure.

One can check:

Cor. 4. $\mathcal{P}(S) \simeq \left\langle (E,\gamma,c) \;\middle|\; \begin{matrix} (E,\gamma) \in \mathcal{B}un_T^W(\tilde{X})(S), \\ c = (c_\alpha)_{\alpha\in\Phi} \text{ as above} \end{matrix} \right\rangle$.

3. Prym varieties

Fix $b \in \mathcal{B}^\circ(k)$.

Recall $\mathrm{Prym}_\Lambda(\tilde{X}_b) := \mathrm{Hom}(\Lambda, \mathrm{Pic}(\tilde{X}_b))^W$,

$\Lambda := \mathrm{Hom}(T, \mathbb{G}_m)$.

Let $\mathrm{Bun}_T^W(\tilde{X}_b) :=$ coarse moduli space of $\mathcal{B}un_T^W(\tilde{X}_b)$.

Forgetting the equivariant structure gives a morphism

$$\mathrm{Bun}_T^W(\tilde{X}_b) \longrightarrow \mathrm{Prym}_\Lambda(\tilde{X}_b)$$
$$(E,\gamma) \longmapsto (\lambda \mapsto E \overset{T,\lambda}{\times} \mathbb{G}_m).$$

<u>Lemma 5.</u> We have an exact sequence

$$H^1(W,T) \to \mathrm{Bun}_T^W(\tilde{X}_b) \to \mathrm{Prym}_\Lambda(\tilde{X}_b) \longrightarrow H^2(W,T).$$

Proof. Equivariant structure γ on $E \in \mathrm{Prym}_\Lambda(\tilde{X}_b)$ amounts to splitting of

$$1 \longrightarrow T \longrightarrow \{(w,\gamma_w) \mid w \in W,\ \gamma_w : w(E) \xrightarrow{\sim} E\} \xrightarrow{pr} W \to 1$$

Such a splitting exists iff the class of this extension in $H^2(W,T)$ vanishes.

Furthermore, the splittings of the trivial extension $T \rtimes W$ form a torsor under $H^1(W,T)$:

To any splitting $s = \gamma \rtimes id_W : W \to T \rtimes W$ attach the class $[(\gamma(w))_{w\in W}] \in H^1(W,T)$.

$\square$

<u>Cor. 6</u> Put $\mathcal{P}_b := \mathscr{P}_b /\!\!/ Z(G)$

$\mathcal{P}^1_b := \mathscr{P}^1_b /\!\!/ Z^1(G)$ where $Z^1(G) := \bigcap_{\alpha \in \Phi} \alpha^{-1}(\mu_{\alpha^\vee})$

$= T^W.$

Then $\mathcal{P}^1_b \overset{\sim}{\hookrightarrow} \mathrm{Prym}_\Lambda(\tilde{X}_b)$

iso on connected cpts

and

$$0 \to \left(\prod_{\alpha \in \Phi} \pi_*(\mu_{\alpha^\vee}, \tilde{X}^\alpha_b)\right)^W \Big/ Z^1(G) \to \mathcal{P}_b \to \mathcal{P}^1_b \to 0$$

is exact.

Proof.

$$\begin{tikzcd}
 & & \left(\prod_{\alpha \in \Phi} \pi_*(T_\alpha, \tilde{X}^\alpha_b)\right)^W & \\
 & H^1(W, T) \arrow[r] \arrow[ur, hook, "\circledast"] & \mathrm{Bun}^W_T(\tilde{X}_b) \arrow[r, "\text{(lemma 5)}"] \arrow[u] & \mathrm{Prym}_\Lambda(\tilde{X}_b) \\
\dfrac{\left(\prod_{\alpha \in \phi} \pi_*(\mu_{\alpha^\vee}, \tilde{X}^\alpha_b)\right)^W}{Z^1(G)} \arrow[r] & \mathcal{P} \arrow[r] & \mathcal{P}^1 \arrow[r, "\text{(prop. 3)}"] \arrow[u] \arrow[ur, hook] & 0 \\
 & & 0 \arrow[u, "\text{(Prop. 1)}"'] &
\end{tikzcd}$$

For $\circledast$ see Hammerli, Matthey, Suter, J. Lie theory 14 (2004), prop. 2.6 (iii).

$\square$